Dual spacetimes, Mach’s Principle and topological defects

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Abstract

By resolving the Riemann curvature relative to a unit timelike vector into electric and magnetic parts, we define a duality transformation which interchanges active and passive electric parts. It implies interchange of roles of Ricci and Einstein curvatures. Further by modifying the vacuum/flat equation we construct spacetimes dual to the Schwarzschild solution and flat spacetime. The dual spacetimes describe the original spacetimes with global monopole charge and global texture. The duality so defined is thus intimately related to the topological defects and also renders the Schwarzschild field asymptotically non-flat which augurs well with Mach’s Principle.

PACS numbers : 04.20,04.60,98.80Hw

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1 Introduction

In analogy with the electromagnetic field, it is possible to resolve the gravitational field; i.e. Riemann curvature tensor into electric and magnetic parts relative to a unit timelike vector [1-2]. In general, a field is produced by charge (source) and its manifestation when charge is stationary is termed as electric and magnetic when it is moving. Electromagnetic field is the primary example of this general feature, which is true for any classical field. In gravitation, unlike other fields, charge is also of two kinds. In addition to the usual charge in terms of non-gravitational energy, gravitational field energy also has charge. Thus electric part would also be of two kinds corresponding to the two kinds of charge, which we term as active and passive.

The Einstein vacuum equation, written in terms of electric and magnetic parts is symmetric in active and passive electric parts. We define the duality relation as interchange of active and passive electric parts. Then it turns out that the Ricci and the Einstein tensors are dual of each other. That is, the non-vacuum equation will in general distinguish between active and passive parts and we could have solutions that are dual of each other [3]. In particular it follows that perfect fluid spacetimes with the equation of states, $\rho - 3p = 0$ and $\rho + p = 0$ are self dual ($\Lambda \rightarrow \Lambda$) while the stiff fluid is dual to dust.

The question is, can we obtain a dual to a vacuum solution? Since the equation is symmetric in active and passive parts, it would remain invariant under the duality transformation. However it turns out that in obtaining the well-known black hole solutions not all of the vacuum equations are used. In particular, for the Schwarzschild solution the equation $R_{00} = 0$ in the standard curvature coordinates is implied by the rest of the equations. If we tamper this equation, the Schwarzschild solution would remain undisturbed for the rest of the set will determine it completely. However this modification, which does not affect the vacuum solution, breaks the symmetry between active and passive electric parts leading to non-invariance of the modified equation under the duality transformation. Now we can have solution dual to vacuum which is different. This is precisely what happens for the Schwarzschild solution.

The Schwarzschild is the unique spherically symmetric vacuum solution, which means it characterizes vacuum for spherical symmetry. It is true that
not all the equations are used in getting to the solution. In fact it turns out that ultimately the equations reduce to the Laplace equation and its first integral [4-5]. That means the Laplace equation becomes free as it would be implied by its first integral equation. Without disturbing the Schwarzschild solution we could introduce some energy density on the right which would be wiped out by the other equations. The modified equation would turn out to be not invariant under the duality transformation, yet however it admits the Schwarzschild solution as the unique solution. Now the dual set of equations also admits the unique solution, which could be interpreted as representing the Schwarzschild particle with global monopole charge [6]. The static black hole with and without global monopole charge are hence dual of each other.

Similarly it turns out that flat spacetime could as well be characterized by a duality non-invariant form of the equation. The static solution of the dual equation will represent massless global monopole (putting the Schwarzschild mass zero in the above solution) and the non-static homogeneous solution will give the FRW metric with the equation of state $\rho + 3p = 0$, which is the characterizing property of global texture [7-8]. The former could as well be looked upon as spacetime of uniform relativistic gravitational potential [4-5]. Global monopoles and textures are stable topological defects which are produced in phase transitions in the early universe when global symmetry is spontaneously broken [7-10]. In particular a global monopole is produced when the global $O(3)$ symmetry is broken into $U(1)$. A solution for a Schwarzschild particle with global monopole charge has been obtained by Barriola and Vilenkin [6]. It therefore follows that the Schwarzschild and the Barriola-Vilenkin solutions are related through the duality transformation. They are dual of each other. Like the Schwarzschild solution, the global monopole solution is also unique. Applications to cosmology and properties of global monopoles [10-14] and of global textures [7-8,11,15-19] have been studied by several authors. What dual solution signifies is restoration of gauge freedom of choosing zero of relativistic potential which was not permitted by the vacuum equation that implied asymptotic flatness. This means that the dual solution breaks asymptotic flatness of the Schwarzschild filed without altering its basic physical character. The relativistic potential is now given by $\phi = k - M/r$ instead of $\phi = -M/r$. This is precisely what is required to make the Schwarzschild field consistent with Mach’s principle. The constant $k$ brings in the information of the rest of the Universe, say for solar
system moving towards the great attractor [20]. The important difference between the Newtonian and relativistic understanding of the problem is that constant $k$ produces non-zero curvature and hence has non-trivial physical meaning. This is the most harmless way of making the field of an isolated body consistent with Mach’s principle.

In sec. 2, we shall give the electromagnetic decomposition of the Riemann curvature, followed by the duality transformation and dual spacetimes in sec. 3 and concluded with discussion in sec. 4.

2 Electromagnetic decomposition

We resolve the Riemann curvature tensor relative to a unit timelike vector [1-2] as follows:

$$E_{ac} = R_{abcd} u^b u^d, \tilde{E}_{ac} = *R_{abcd} u^b u^d$$  \hspace{1cm} (1)

$$H_{ac} = *R_{abcd} u^b u^d = H_{(ac)} - H_{[ac]}$$  \hspace{1cm} (2)

where

$$H_{(ac)} = *C_{abcd} u^b u^d$$  \hspace{1cm} (3)

$$H_{[ac]} = \frac{1}{2} \eta_{abce} R^e_{ab} u^b u^d.$$  \hspace{1cm} (4)

Here $C_{abcd}$ is the Weyl conformal curvature, $\eta_{abce}$ is the 4-dimensional volume element. $E_{ab} = E_{ba}, \tilde{E}_{ab} = \tilde{E}_{ba}, (E_{ab}, \tilde{E}_{ab}, H_{ab}) u^b = 0$, $H = H^a_a = 0$ and $u^a u_a = 1$. The Ricci tensor could then be written as

$$R_{ab} = E_{ab} + \tilde{E} - ab + (E + \tilde{E}) u_a u_b - \tilde{E} g_{ab} + \frac{1}{2} H^{mn} u^c (\eta_{acmn} u_b + \eta_{bcmn} u_a)$$  \hspace{1cm} (5)

where $E = E^a_a$ and $\tilde{E} = \tilde{E}^a_a$. It may be noted that $E = (\tilde{E} + \frac{1}{2} T)/2$ defines the gravitational charge density while $\tilde{E} = -T_{ab} u^a u^b$ defines the energy density relative to the unit timelike vector $u^a$. 

4
3 Duality transformation and dual spacetimes

The vacuum equation, $R_{ab} = 0$ is in general equivalent to

$$E \text{ or } \tilde{E} = 0, \ H_{[ab]} = 0 = E_{ab} + \tilde{E}_{ab}$$

which is symmetric in $E_{ab}$ and $\tilde{E}_{ab}$.

We define the duality transformation as

$$E_{ab} \longleftrightarrow \tilde{E}_{ab}, \ H_{[ab]} = H_{[ab]}.$$  \hfill (7)

Thus the vacuum equation (6) is invariant under the duality transformation (7). From eqn. (1) it is clear that the duality transformation would map the Ricci tensor into the Einstein tensor and vice-versa. This is because contraction of Riemann is Ricci while of its double dual is Einstein.

Consider the spherically symmetric metric,

$$ds^2 = c^2(r,t)dt^2 - a^2(r,t)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2).$$  \hfill (8)

The natural choice for the resolving vector in this case is of course it being hypersurface orthogonal, pointing along the $t$-line. From eqn. (6), $H_{[ab]} = 0$ and $E_{2}^2 + \tilde{E}_{2}^2 = 0$ lead to $ac = 1$ (for this, no boundary condition of asymptotic flatness need be used). Now $\tilde{E} = 0$ gives $a = (1 - 2M/r)^{-1/2}$, which is the Schwarzschild solution. Note that we did not need to use the remaining equation and $E_{1}^1 + \tilde{E}_{1}^1 = 0$, it is hence free and is implied by the rest. Without affecting the Schwarzschild solution, we can introduce some distribution in the 1-direction.

We hence write the alternate equation as

$$H_{[ab]} = 0 = \tilde{E}, \ E_{ab} + \tilde{E}_{ab} = \lambda w_{a}w_{b}$$  \hfill (9)

where $\lambda$ is a scalar and $w_{a}$ is a spacelike unit vector along 4-acceleration. It is clear that it will also admit the Schwarzschild solution as the general solution, and it determines $\lambda = 0$. That is for spherical symmetry the above
alternate equation also characterizes vacuum, because the Schwarzschild solution is unique.

Let us now employ the duality transformation (7) to the above equation (9) to write

$$ H_{[ab]} = 0 = E, \ E_{ab} + \bar{E}_{ab} = \lambda w_a w_b. \tag{10} $$

Its general solution for the metric (8) is given by

$$ c = a^{-1} = (1 - 2k - \frac{2M}{r})^{1/2}. \tag{11} $$

This is the Barbiola-Vilenkin solution [6] for the Schwarzschild particle with global monopole charge, $\sqrt{2k}$. Again we shall have $ac = 1$ and $E = 0$ will then yield $c = (1 - 2k - 2M/r)^{1/2}$ and $\lambda = 2k/r^2$. This has non-zero stresses given by

$$ T^0_0 = T^1_1 = \frac{2k}{r^2}. \tag{12} $$

A global monopole is described by a triplet scalar, $\psi^a(r) = \eta \psi(r)x^a/r, x^a x^a = r^2$, which through the usual Lagrangian generates energy-momentum distribution at large distance from the core precisely of the form given above in (12) [6]. Like the Schwarzschild solution the monopole solution (11) is also the unique solution of eqn.(10).

If we translate eqns. (9) and (10) in terms of the familiar Ricci components, they would read as

$$ R^0_0 = R^1_1 = \lambda, R^2_2 = 0 = R_{01} \tag{13} $$

and

$$ R^0_0 = R^1_1 = 0 = R_{01}, R^2_2 = \lambda. \tag{14} $$

For the metric (8), we shall then have $ac = 1$ and $c^2 = f(r) = 1 + 2\phi$, say, and

$$ \kappa^0_0 = - \nabla^2 \phi \tag{15} $$
\[ R_2^2 = -\frac{2}{r^2} (r\phi)' \]  

Now the set (13) integrates to give \( \phi = -M/r \) and \( \lambda = 0 \), which is the Schwarzschild solution while (14) will give \( \phi = -k - M/r \) and \( \lambda = 2k/r^2 \), the Schwarzschild with global monopole charge. Thus global monopole owes its existence to the constant \( k \), appearing in the solution of the usual Laplace equation implied by eqns. (14) and (15). It defines a pure gauge for the Newtonian theory, which could be chosen freely; while the Einstein vacuum equation determines it to be zero. For the dual-vacuum equation (14), it is free like the Newtonian case but it produces non-zero curvature and hence would represent non-trivial physical and dynamical effect (see \( R_2^2 = -2k/r^2 \neq 0 \) unless \( k = 0 \)). This is the crucial difference between the Newtonian theory and GR in relation to this problem, that the latter determines the relativistic potential \( \phi \) absolutely, vanishing only at infinity. This freedom is restored in the dual-vacuum equation, of course at the cost of introducing stresses that represent a global monopole charge. The uniform potential would hence represent a massless global monopole (\( M = 0 \) in the solution (11)), which is solely supported by the passive part of electric field. It has been argued and demonstrated [5] that it is the non-linear aspect of the field (which incorporates interaction of gravitational field energy density) that produces space-curvatures and consequently the passive electric part. It is important to note that the relativistic potential \( \phi \) plays the dual role of the Newtonian potential as well as the non-Newtonian role of producing curvature in space. The latter aspect persists even when potential is constant different from zero. It is the dual-vacuum equation that uncovers this aspect of the field.

On the other hand, flat spacetime could also in alternative form be characterized by

\[ \tilde{E}_{ab} = 0 = H_{[ab]}, E_{ab} = \lambda w_a w_b \]  

leading to \( c = a = 1 \), and implying \( \lambda = 0 \). Its dual will be

\[ E_{ab} = 0 = H_{[ab]}, \tilde{E}_{ab} = \lambda w_a w_b \]  

yielding the general solution,

\[ c' = a' = 0 \quad \Rightarrow \quad c = 1, \ a = \text{const.} = (1 - 2k)^{-1/2} \]
which is non-flat and represents a global monopole of zero mass, as it follows from the solution (11) when \( M = 0 \). This is also the uniform relativistic potential solution.

Further it is known that the equation of state \( \rho + 3p = 0 \) which means \( E = 0 \), characterizes global texture \([7,19]\). That is, the necessary condition for spacetimes of topological defects; global textures and monopoles is \( E = 0 \). Like the uniform potential spacetime, it can also be shown that the global texture spacetime is dual to flat spacetime. In the above eqns (13) and (14), replace \( w_a w_b \) by the projection tensor \( h_{ab} = g_{ab} - u_a u_b \). Then non-static homogeneous solution of the dual-flat equation (14) is the FRW metric with \( \rho + 3p = 0 \), which determines the scale factor \( S(t) = \alpha t + \beta \), and \( \rho = 3(\alpha^2 + k) / (\alpha t + \beta)^2 \), \( k = \pm 1, 0 \). This is also the unique non-static homogeneous solution. The general solutions of the dual-flat equation are thus the massless global monopole (uniform potential) spacetime in the static case and the global texture spacetime in the non-static homogeneous case. Thus they are dual to flat spacetime.

It turns out that spacetimes with \( E = 0 \) can be generated by considering a hypersurface in 5-dimensional Minkowski space defined, for example, by

\[
t^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = k^2(t^2 - x_1^2 - x_2^2 - x_3^2)
\]

(20)
which consequently leads to the metric

\[
d s^2 = k^2 d T^2 - T^2[ d \chi^2 + \sin h^2 \chi (d \theta^2 + \sin^2 \theta d \phi^2) ]
\]

(21)
Here \( T^2 = t^2 - x_1^2 - x_2^2 - x_3^2 \) and \( \rho = 3(1 - k^2)/k^2 T^2 \). The above construction will generate spacetimes of global monopole, cosmic strings (and their homogeneous versions as well), and global texture-like depending upon the dimension and character of the hypersurface. Of course, \( E = 0 \) always; i.e. zero gravitational mass \([11]\). The trace of active part measures the gravitational charge density, responsible for focussing of congruence in the Raychaudhuri equation \([21]\). The topological defects are thus characterized by vanishing of focussing density (tracelessness of active part).

Application of the duality transformation, apart from vacuum/flat case considered here, has been considered for fluid spacetime \([3]\). The duality trans-
formation could similarly be considered for electrovac equation including the \( \Lambda \)-term. Here the analogue of the master equation (10) is

\[
H_{[ab]} = 0, \quad E = \Lambda - \frac{Q^2}{2r^4}, \quad E^a_a + \tilde{E}^b_a = (-\frac{Q^2}{r^4} + \lambda) w_a w^b
\]  

(22)

which has the general solution \( c^2 = a^{-2} = (1 - 2k - 2M/r + Q^2/2r^2 - \Lambda r^2/3) \) and \( \lambda = 2k/r^2 \). The analogue of eqn. (6) will have \( \tilde{E} = -\Lambda - Q^2/2r^4 \) instead of \( E \) in (20). Thus the duality transformation works in general for a charged particle in the de Sitter universe. Similarly spacetime dual to the NUT solution has been obtained [22]. In the case of the Kerr solution it turns out, in contrast to others, that dual solution is not unique. The dual equation admits two distinct solutions which include the original Kerr solution [23].

4 Discussion

First of all let us try to get some physical feel of active, passive and magnetic parts. For a canonical resolution relative to a hypersurface orthogonal unit timelike vector, it follows that \( E_{ab} \) would refer to the curvature components \( R_{0a0a} \), \( \tilde{E}_{ab} \) to \( R_{abab} \) and \( H_{ab} \) to \( R_{0aab} \). With reference to the spherically symmetric metric (8), it can be easily seen that the active part is crucially anchored onto the Newtonian potential appearing in \( g_{90} = 1 + 2\phi \), while the passive part to the relativistic potential, \( g_{11} = -(1 + 2\phi)^{-1} \). Note that in obtaining the Schwarzschild solution we ultimately solve the Laplace equation, which does not take into account contribution of gravitational field energy as source. It can be shown that contribution of gravitational field energy goes into curving the space through \( g_{11} \neq 1 \) leaving the Laplace equation undisturbed [4-5]. Thus passive part is created by the field energy while the active by non-gravitational energy distribution. The magnetic part would as expected be due to motion of sources.

Under the duality transformation, the vacuum equation remains invariant leading to the same solutions, but the Weyl tensor changes sign which would mean \( GM \to -GM \). This happens because active part is produced by positive non-gravitational energy while passive part by negative field energy, and the interchange of active and passive would therefore imply interchange
of positive energy and negative field energy [2].

Consider the Maxwell like duality $E \rightarrow H, H \rightarrow -E$ as given by

$$E_{ab} \rightarrow H_{ab}, \quad H_{ab} \rightarrow -\tilde{E}_{ab}, \quad \tilde{E}_{ab} \rightarrow -E_{ab} \quad (23)$$

This implies $E = 0, H_{[ab]} = 0, E_{ab} + \tilde{E}_{ab} = 0$ which is the vacuum equation (6) and keeps the Einsten action invariant because $R = 2(E - \tilde{E})$. This is a remarkable result indicating that vacuum equation is implied by the duality symmetry of the action [2]. Note also that duality transformation of the action does not permit the cosmological constant which could however be brought in as matter with the specific equation of state. This result is similar to the well-known property of GR that equation of motion for free particle is contained in the field equation.

The duality transformation (7) introduces in most harmless manner a global monopole in the Schwarzschild black hole which amounts to breaking the asymptotic flatness. The latter is a necessary requirement for the field to be consistent with Mach’s principle at the very elementary level. In essence, it is obtained by simply retaining the constant of integration in the solution of the Laplace equation. Thus it makes no difference at the Newtonian level and hence its contribution is purely relativistic.

The most general duality-invariant expression consisting of the Ricci and the metric is $R^a_b - (R^b_a - \Lambda)g^a_b$. This, without $\Lambda$ equal to zero would be the equation for gravitational instanton, which follows from the $R^2$-action. The instanton action and the field equation are duality-invariant. They are also conformally invariant as well. As a matter of fact conformal invariance singles out the $R^2$-instanton action. That means the conformal invariance includes the duality invariance, while the duality invariance of the Palatini action with the condition that the resulting equation be valid for all values of $R$ would lead to the conformal invariance [24-25]. The simplest and well-known instanton solution is the de Sitter spacetime. Here the duality only leads to the anti-de Sitter.

Acknowledgement : I have pleasure in thanking Jose Senovilla, and LK Patel and Ramesh Tikekar for useful discussions. Above all it is a matter of
great pleasure and privilege to having known Jayant and worked with him closely in setting up IUCAA. With deep affection and feeling I dedicate this work to him on his completing 60.
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