The scalar bi-spectrum in the Starobinsky model

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A (partial?) list of ever-increasing number of inflationary models. May be, we should look for models that permit deviations from the standard picture of slow roll inflation.

Based on

Outline of the talk

1. Features, fits and non-Gaussianities
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2. The scalar power spectrum in the Starobinsky model
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3. The non-Gaussianity parameter $f_{NL}$
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2. The scalar power spectrum in the Starobinsky model
3. The non-Gaussianity parameter $f_{\text{NL}}$
4. The method for evaluating $f_{\text{NL}}$
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1. Features, fits and non-Gaussianities
2. The scalar power spectrum in the Starobinsky model
3. The non-Gaussianity parameter $f_{NL}$
4. The method for evaluating $f_{NL}$
5. $f_{NL}$ in the Starobinsky model (in the equilateral limit)
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1. Features, fits and non-Gaussianities
2. The scalar power spectrum in the Starobinsky model
3. The non-Gaussianity parameter $f_{NL}$
4. The method for evaluating $f_{NL}$
5. $f_{NL}$ in the Starobinsky model (in the equilateral limit)
6. Summary
The WMAP 7-year data for the CMB TT angular power spectrum (the black dots with error bars) and the theoretical, best fit $\Lambda$CDM model with a power law primordial spectrum (the solid red curve). Note the outliers near the multipoles $\ell = 2, 22$ and 40.

Reconstructing the primordial spectrum

Reconstructed primordial spectra, obtained upon assuming the concordant background $\Lambda$CDM model. The recovered spectrum on the left improves the fit to the WMAP 3-year data by $\Delta \chi^2_{\text{eff}} \approx 15$, with respect to the best fit power law spectrum$^3$. The spectrum on the right has been recovered from a variety of CMB datasets, including the WMAP 5-year data$^4$.

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The scalar power spectra in a few different inflationary models that lead to a better fit to the CMB data than the conventional power law spectrum.

'Large’ non-Gaussianities and its possible implications

- The WMAP 7-year data constrains the non-Gaussianity parameter $f_{NL}$ to be $f_{NL} = (26 \pm 140)$ in the equilateral limit, at 68% confidence level\(^6\).

- If forthcoming missions such as Planck detect a large level of non-Gaussianity, as suggested by the above mean value of $f_{NL}$, then it can result in a substantial tightening in the constraints on the various inflationary models. For example, canonical scalar field models that lead to a nearly scale invariant primordial spectrum contain only a small amount of non-Gaussianity and, hence, will cease to be viable\(^7\).

- However, it is known that primordial spectra with features can lead to reasonably large non-Gaussianities\(^8\). Therefore, if the non-Gaussianity parameter $f_{NL}$ indeed proves to be large, then either one has to reconcile with the fact that the primordial spectrum contains features or we have to turn our attention to non-canonical scalar field models such as, say, D brane inflation models\(^9\).

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\(^7\) J. Maldacena, JHEP 05, 013 (2003).
\(^8\) See, for instance, X. Chen, R. Easther and E. A. Lim, JCAP 0706, 023 (2007).
\(^9\) See, for example, X. Chen, M.-x. Huang, S. Kachru and G. Shiu, JCAP 0701, 002 (2007).
The Starobinsky model involves the canonical scalar field which is described by the potential

\[ V(\phi) = \begin{cases} 
V_0 + A_+ (\phi - \phi_0) & \text{for } \phi > \phi_0, \\
V_0 + A_- (\phi - \phi_0) & \text{for } \phi < \phi_0.
\end{cases} \]

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Assumptions and properties

- It is assumed that the constant $V_0$ is the dominant term in the potential for a range of $\phi$ near $\phi_0$. As a result, over the domain of our interest, the expansion is of the de Sitter form corresponding to a Hubble parameter $H_0$ determined by $V_0$.

- The scalar field rolls slowly until it reaches the discontinuity in the potential. It then fast rolls for a brief period as it crosses the discontinuity before slow roll is restored again.

- Since $V_0$ is dominant, the first slow roll parameter $\epsilon_1$ remains small even during the transition. This property allows the background to be evaluated analytically to a good approximation.
Analytic expressions for the slow roll parameters

Under the assumptions and approximations described above, the slow roll parameters remain small before the transition.
Analytic expressions for the slow roll parameters

Under the assumptions and approximations described above, the slow roll parameters remain small before the transition.

One can show that, after the transition, the evolution of the first slow roll parameter $\epsilon_1$ can be expressed in terms of the number of e-folds $N$ as follows:

$$
\epsilon_1 \sim \frac{A_-^2}{18 \, M_{Pl}^2 \, H_0^4} \left[ 1 - \frac{\Delta A}{A_-} \, e^{-3 (N - N_0)} \right]^2,
$$

where $\Delta A = (A_- - A_+)$, while $N_0$ is the e-fold at which the field crosses the discontinuity.
Analytic expressions for the slow roll parameters

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One can show that, after the transition, the evolution of the first slow roll parameter $\epsilon_1$ can be expressed in terms of the number of e-folds $N$ as follows:

$$\epsilon_1^- \simeq \frac{A_-^2}{18 M_{Pl}^2 H_0^4} \left[ 1 - \frac{\Delta A}{A_-} e^{-3 (N-N_0)} \right]^2,$$

where $\Delta A = (A_- - A_+)$, while $N_0$ is the e-fold at which the field crosses the discontinuity.

It is found that, \textit{immediately after the transition}, the second slow roll parameter $\epsilon_2$ is given by

$$\epsilon_2^- \simeq \frac{6 \Delta A}{A_-} \frac{e^{-3 (N-N_0)}}{1 - (\Delta A/A_-) e^{-3 (N-N_0)}}.$$
The evolution of the first slow roll parameter $\epsilon_1$ on the left, and the second slow roll parameter $\epsilon_2$ on the right in the Starobinsky model. While the blue curves describe the numerical results, the dotted red curves represent the analytical expressions mentioned in the previous slide.
The scalar power spectrum in the Starobinsky model

Evolution of the perturbations

The modes before and after the transition

It can be shown that, under the assumptions that one is working with, the quantity \( z = a M_{P1} \sqrt{2 \epsilon_1} \), which determines the evolution of the perturbations, simplifies to

\[
\frac{z''}{z} \simeq 2 H^2
\]

both before as well as after the transition with the overprime denoting the derivative with respect to the conformal time, while \( H \) is the conformal Hubble parameter.
It can be shown that, under the assumptions that one is working with, the quantity $z = a M_{\text{Pl}} \sqrt{2 \epsilon_1}$, which determines the evolution of the perturbations, simplifies to

$$z'' / z \simeq 2 \mathcal{H}^2$$

both before as well as after the transition with the overprime denoting the derivative with respect to the conformal time, while $\mathcal{H}$ is the conformal Hubble parameter.

As a result, while the solution to the Mukhanov-Sasaki variable $v_k$ before the transition is given by

$$v_k^+ (\eta) = \frac{1}{\sqrt{2 k}} \left(1 - \frac{i}{k \eta}\right) e^{-i k \eta},$$

after the transition, it can be expressed as a linear combination of the positive and the negative frequency modes as follows:

$$v_k^- (\eta) = \frac{\alpha_k}{\sqrt{2 k}} \left(1 - \frac{i}{k \eta}\right) e^{-i k \eta} + \frac{\beta_k}{\sqrt{2 k}} \left(1 + \frac{i}{k \eta}\right) e^{i k \eta},$$

where $\alpha_k$ and $\beta_k$ are the usual Bogoliubov coefficients.
The scalar power spectrum in the Starobinsky model

The Bogoliubov coefficients $\alpha_k$ and $\beta_k$ can be obtained by matching the mode $v_k$ and its derivative at the transition.
The scalar power spectrum in the Starobinsky model

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The scalar power spectrum, given by

$$P_S(k) = (k^3/2 \pi^2) |R_k|^2 = (k^3/2 \pi^2) (|v_k|/z)^2,$$

where $R_k$ is the curvature perturbation, can be evaluated at late times to be

$$P_S(k) = \left( \frac{9 H_0^6}{4 \pi^2 A_-^2} \right) \left\{ 1 - \frac{3 \Delta A k_0}{A_+ k} \left[ \left(1 - \frac{k_0^2}{k^2} \right) \sin \left(\frac{2 k}{k_0}\right) + \frac{2 k_0}{k} \cos \left(\frac{2 k}{k_0}\right) \right] + \frac{9 \Delta A^2 k_0^2}{2 A_+^2 k^2} \left(1 + \frac{k_0^2}{k^2} \right) \left[ \left(1 + \frac{k_0^2}{k^2} \right) - \frac{2 k_0}{k} \sin \left(\frac{2 k}{k_0}\right) \right] + \left(1 - \frac{k_0^2}{k^2} \right) \cos \left(\frac{2 k}{k_0}\right) \right\},$$

where $k_0$ is the wavenumber of the mode that crosses the Hubble radius when the field crosses the discontinuity. Note that the power spectrum depends on the wavenumber only through the ratio $(k/k_0)$. 

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The scalar bi-spectrum in the Starobinsky model

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The scalar power spectrum in the Starobinsky model. While the blue solid curve denotes the analytic result, the red dots represent the corresponding numerical scalar power spectrum that has been obtained through an exact integration of the background as well as the perturbations.
The scalar bi-spectrum

The scalar bi-spectrum $B_S(k_1, k_2, k_3)$ is related to the three point correlation function of the Fourier modes of the curvature perturbation, evaluated towards the end of inflation, say, at the conformal time $\eta_e$, as follows\textsuperscript{11}:

$$\langle \hat{R}_{k_1}(\eta_e) \hat{R}_{k_2}(\eta_e) \hat{R}_{k_3}(\eta_e) \rangle = (2\pi)^3 B_S(k_1, k_2, k_3) \delta^3(k_1 + k_2 + k_3).$$

For convenience, we shall set

$$B_S(k_1, k_2, k_3) = (2\pi)^{-9/2} G(k_1, k_2, k_3).$$

The observationally relevant non-Gaussianity parameter $f_{\text{NL}}$ is introduced through the equation\textsuperscript{12}

$$
\mathcal{R}(\eta, x) = \mathcal{R}^G(\eta, x) - \frac{3}{5} f_{\text{NL}} [\mathcal{R}^G(\eta, x)]^2,
$$

where $\mathcal{R}^G$ denotes the Gaussian quantity, and the factor of $(3/5)$ arises due to the relation between the Bardeen potential and the curvature perturbation during the matter dominated epoch.

\textsuperscript{12} J. Maldacena, JHEP \textbf{0305}, 013 (2003); S. Hannestad, T. Haugbolle, P. R. Jarnhus and M. S. Sloth, JCAP \textbf{1006}, 001 (2010).
The introduction of $f_{NL}$

The observationally relevant non-Gaussianity parameter $f_{NL}$ is introduced through the equation\(^\text{12}\)

$$
\mathcal{R}(\eta, x) = \mathcal{R}^G(\eta, x) - \frac{3 f_{NL}}{5} \left[ \mathcal{R}^G(\eta, x) \right]^2,
$$

where $\mathcal{R}^G$ denotes the Gaussian quantity, and the factor of $(3/5)$ arises due to the relation between the Bardeen potential and the curvature perturbation during the matter dominated epoch.

In Fourier space, the above equation can be written as

$$
\mathcal{R}_k = \mathcal{R}^G_k - \frac{3 f_{NL}}{5} \int \frac{d^3 p}{(2 \pi)^{3/2}} \mathcal{R}^G_p \mathcal{R}^G_{k-p}.
$$

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\(^{12}\) J. Maldacena, JHEP **0305**, 013 (2003);
S. Hannestad, T. Haugbolle, P. R. Jarnhus and M. S. Sloth, JCAP **1006**, 001 (2010).
Using this relation and Wick’s theorem, one can arrive at the three point correlation of the curvature perturbation in Fourier space in terms of the parameter $f_{NL}$. It is found to be

$$
\langle \mathcal{R}_{k_1} \mathcal{R}_{k_2} \mathcal{R}_{k_3} \rangle = - \left( \frac{3 f_{NL}}{10} \right) (2\pi)^4 (2\pi)^{-3/2} \left( \frac{1}{k_1^3 k_2^3 k_3^3} \right) \delta^{(3)}(k_1 + k_2 + k_3) \\
\times \left[ k_1^3 \mathcal{P}_S(k_2) \mathcal{P}_S(k_3) + \text{two permutations} \right].
$$
The relation between $f_{NL}$ and the bi-spectrum

Using the above expression for the three point function of the curvature perturbation and the definition of the bi-spectrum, we can then arrive at the following relation:

$$f_{NL} = - \left( \frac{10}{3} \right) (2\pi)^{-4} (2\pi)^{9/2} \left( k_1^3 k_2^3 k_3^3 \right) \mathcal{B}_S(k_1, k_2, k_3)$$

$$\times \left[ k_1^3 \mathcal{P}_S(k_2) \mathcal{P}_S(k_3) + \text{two permutations} \right]^{-1}$$

$$= - \left( \frac{10}{3} \right) (2\pi)^{-4} \left( k_1^3 k_2^3 k_3^3 \right) G(k_1, k_2, k_3)$$

$$\times \left[ k_1^3 \mathcal{P}_S(k_2) \mathcal{P}_S(k_3) + \text{two permutations} \right]^{-1}.$$
The relation between $f_{\text{NL}}$ and the bi-spectrum

Using the above expression for the three point function of the curvature perturbation and the definition of the bi-spectrum, we can then arrive at the following relation:

$$f_{\text{NL}} = -\left(\frac{10}{3}\right) (2\pi)^{-4} \left(2\pi\right)^{9/2} \left(k_1^3 k_2^3 k_3^3\right) \mathcal{B}_s(k_1, k_2, k_3) \times \left[k_1^3 \mathcal{P}_s(k_2) \mathcal{P}_s(k_3) + \text{two permutations}\right]^{-1}$$

$$= -\left(\frac{10}{3}\right) (2\pi)^{-4} \left(k_1^3 k_2^3 k_3^3\right) \mathcal{G}(k_1, k_2, k_3) \times \left[k_1^3 \mathcal{P}_s(k_2) \mathcal{P}_s(k_3) + \text{two permutations}\right]^{-1}.$$

In the equilateral limit (i.e. when $k_1 = k_2 = k_3$), this expression for $f_{\text{NL}}$ simplifies to

$$f_{\text{NL}} = -\left(\frac{10}{9}\right) (2\pi)^{-4} \left(\frac{k^6 \mathcal{G}(k)}{\mathcal{P}_s^2(k)}\right).$$
The action at the cubic order\(^\text{13}\)

It can be shown that the third order term in the action describing the curvature perturbations is given by

\[
S_3[\mathcal{R}] = M_{\text{Pl}}^2 \int d\eta \int d^3x \left[ a^2 \epsilon_1^2 \mathcal{R} \mathcal{R}'^2 + a^2 \epsilon_1^2 \mathcal{R} (\partial \mathcal{R})^2 - 2a \epsilon_1 \mathcal{R}' (\partial^i \mathcal{R})(\partial_i \chi) + \frac{a^2}{2} \epsilon_1 \epsilon_2 \mathcal{R}^2 \mathcal{R}' + \frac{\epsilon_1}{2} (\partial^i \mathcal{R})(\partial_i \chi)(\partial^2 \chi) + \frac{\epsilon_1}{4} (\partial^2 \mathcal{R})(\partial \chi)^2 + \mathcal{F}\left(\frac{\delta L_2}{\delta \mathcal{R}}\right)\right],
\]

where \(\mathcal{F}(\delta L_2/\delta \mathcal{R})\) denotes terms involving the variation of the second order action with respect to \(\mathcal{R}\), while \(\chi\) is related to the curvature perturbation \(\mathcal{R}\) through the relations

\[
\Lambda = a \epsilon_1 \mathcal{R}' \quad \text{and} \quad \partial^2 \chi = \Lambda.
\]

\(^{13}\) J. Maldacena, JHEP 0305, 013 (2003); D. Seery and J. E. Lidsey, JCAP 0506, 003 (2005); X. Chen, M.-x. Huang, S. Kachru and G. Shiu, JCAP 0701, 002 (2007).
Evaluating the bi-spectrum

At the leading order in the perturbations, one then finds that the three point correlation in Fourier space is described by the integral

$$\langle \hat{R}_{k_1}(\eta_e) \hat{R}_{k_2}(\eta_e) \hat{R}_{k_3}(\eta_e) \rangle = -i \int_{\eta_i}^{\eta_e} d\eta \ a(\eta) \left\langle \left[ \hat{R}_{k_1}(\eta_e) \hat{R}_{k_2}(\eta_e) \hat{R}_{k_3}(\eta_e), \hat{H}_I(\eta) \right] \right\rangle,$$

where $\hat{H}_I$ is the operator corresponding to the above third order action, while $\eta_i$ is the time at which the initial conditions are imposed on the modes when they are well inside the Hubble radius, and $\eta_e$ denotes a very late time, say, close to when inflation ends.

In the equilateral limit, the quantity $G(k)$, evaluated towards the end of inflation at the conformal time $\eta = \eta_e$, can be written as

$$G(k) \equiv \sum_{C=1}^{6} G_C(k) = M_{Pl}^2 \sum_{C=1}^{6} \left[ f_k^3(\eta_e) G_C(k) + f_k^3(\eta_e) G_C^*(k) \right],$$

where the quantities $G_C(k)$ are integrals that correspond to six terms that arise in the action at the third order in the perturbations, while $f_k$ are the modes associated with the curvature perturbation $R_k$. 
When there exist deviations from slow roll, it is found that the fourth term $G_4$ provides the dominant contribution to $f_{\text{NL}}$.

It is described by the following integral

$$G_4(k) = 3i \int_{\eta_i}^{\eta_e} d\eta \ a^2 \epsilon_1' \epsilon_2 f_k^* f_k'.$$

In the case of the Starobinsky model, as $\epsilon_2$ is a constant before the transition, $\epsilon_2'$ vanishes, and hence the above integral $G_4$ is non-zero only post-transition.
Evaluating $f_{\text{NL}}$ in the Starobinsky model

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It is described by the following integral

$$G_4(k) = 3i \int_{\eta_i}^{\eta_e} d\eta \ a^2 \epsilon_2 \epsilon'_2 \ f^*_2 \ f'_2.$$ 

In the case of the Starobinsky model, as $\epsilon_2$ is a constant before the transition, $\epsilon'_2$ vanishes, and hence the above integral $G_4$ is non-zero only post-transition.

We find that the integral involved can be computed analytically.
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$$G_4(k) = 3 i \int_{\eta_i}^{\eta_e} d\eta \ a^2 \epsilon_1 \epsilon_2' f_k^* f_k'^*.$$ 

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In fact, with some effort, analytic expressions can be arrived at for all the $G_n$. 

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The dominant contribution in the Starobinsky model

The absolute value of the quantity \( [k^6 G_4] \) has been plotted as a function of \( (k/k_0) \) (the blue curve). We have worked with the same of values of \( A_+ \), \( A_- \) and \( V_0 \) as in the earlier figure wherein we had plotted the power spectrum. The green and the red curves in the inset represent the limiting values for \( k \ll k_0 \) and \( k \gg k_0 \), respectively.
The different contributions

The quantities $k^6$ times the absolute values of $(G_1 + G_3)$ (in green), $G_2$ (in red), $G_4$ (in blue) and $(G_5 + G_6)$ (in purple) have been plotted as a function of $(k/k_0)$ for the Starobinsky model.
The non-Gaussianity parameter $f_{NL}$ due to the dominant term in the Starobinsky model, plotted as a function of $(k/k_0)$ for $(A_-/A_+) = 0.216$ and $(A_-/A_+) = 0.0216$. Larger the difference between $A_-$ and $A_+$, larger is the corresponding $f_{NL}$.
The non-Gaussianity parameter $f_{\text{NL}}$ due to the dominant term in the Starobinsky model, plotted as a function of $(k/k_0)$ and the ratio $r = (A_-/A_+)$. The white contours indicate regions wherein $f_{\text{NL}}$ can be as large as 50. Note that, provided $r$ is reasonably small, $f_{\text{NL}}$ can be of the order of 20 or so, as is indicated by the currently observed mean value.
Amazingly, we find that, for a certain range of values of the parameters involved, the non-Gaussianity parameter $f_{NL}$ can be evaluated analytically, to a good accuracy (as is confirmed by comparison with numerical computations) in the Starobinsky model.
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Interestingly, for suitably small values of $r = (A_- / A_+)$, $f_{NL}$ in the Starobinsky model can be as large as indicated by the currently observed mean values.
Thank you for your attention