Statistics of Statistical Anisotropy Measures

Nidhi Joshi
Centre for Theoretical Physics
Jamia Millia Islamia

Collaborators: Aditya Rotti, Tarun Souradeep

Confronting particle-cosmology with Planck and LHC
August 10-12, 2011
Correlation is a two point function & can be expanded in bipolar spherical harmonics basis.

\[ C(\hat{n}_1, \hat{n}_2) = \sum_{l_1,l_2,L,M} A_{l_1l_2}^{LM} \{ Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2) \}_L^M \]

\[ A_{l_1l_2}^{LM} \rightarrow \text{Bipolar spherical harmonic (BipoSH) coefficients} \]

\[ \{ Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2) \}_L^M \rightarrow \text{Bipolar spherical harmonics} \]

Convenient basis of expansion for functions depending on two vector directions

\[ \{ Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2) \}_L^M = \sum_{m_1m_2} C_{l_1m_1l_2m_2}^{LM} Y_{l_1m_1}(\hat{n}_1) Y_{l_2m_2}(\hat{n}_2) \]

\[ |l_1 - l_2| \leq L \leq l_1 + l_2 \quad \text{Triangularity conditions} \]

Orthonormal functions for different set of \( l_1, l_2, L, M \)
Statistical Isotropy in bipolar space

\[ C(\hat{n}_1, \hat{n}_2) = C(\hat{n}_1 \cdot \hat{n}_2) = C(\Theta) \]

Correlation function is invariant under the rotations

\[ L = 0, M = 0, l_1 = l_2 \]

Any Statistical isotropy violation signal can be searched for in BipoSH coefficients!!

Search for \( L \neq 0 \) BipoSH coefficients

Detection of SI violation

The quadrupolar bipolar power spectra, binned with $l = 50$, using the KQ75y7 mask.

Error Bars?? Is distribution symmetric??
Understanding is extremely crucial to assess the significance of any statistical isotropy violation detection!!
Characteristic Function Approach....

Characteristic function is fourier transform of the density function,

$$\varphi_Z(t) = E[e^{itZ}]$$

If $X_1, X_2, ..., X_n$ are independent random variables and $a_1, a_2, ..., a_n$ are some constants.

The characteristic function of linear combination of $X_i$'s ($Z = \sum_i^n a_i X_i$) is,

$$\varphi_{Z_n}(t) = \varphi_{X_1}(a_1 t) \varphi_{X_2}(a_2 t) \cdots \varphi_{X_n}(a_n t)$$

Cumulant generating function,

$$g_Z(t) = \log[\varphi_Z(t)]$$

**Cumulants**

$$K_n = i^n g^n_Z(t) \bigg|_{t=0}$$
Relationship between Cumulants & Moments

\[
\begin{align*}
\mu_1 &= K_1, \\
\mu_2 &= K_2, \\
\mu_3 &= K_3, \\
\mu_4 &= K_4 + 3K_2^2, \\
\mu_5 &= K_5 + 10K_3K_2, \\
\mu_6 &= K_6 + 15K_4K_2 + 10K_3^2 + 15K_2^3.
\end{align*}
\]

Normalized Moments

\[
\mu_{n, \text{norm}} = \frac{\mu_n}{\sigma^n}
\]
BipoSH coefficients can also be expressed as linear combination of random variables,

\[ A_{l_1 l_2}^{LM} = \sum_{m_1 m_2} a_{l_1 m_1} a_{l_2 m_2} C_{l_1 m_1 l_2 m_2}^{LM} \]

Real and Imaginary parts of BipoSH coefficients,

\[ A_{l_1 l_2}^{LM(R)} = \sum_{m_1 m_2} (x_{l_1 m_1} x_{l_2 m_2} - y_{l_1 m_1} y_{l_2 m_2}) C_{l_1 m_1 l_2 m_2}^{LM} \]

\[ A_{l_1 l_2}^{LM(I)} = \sum_{m_1 m_2} (y_{l_1 m_1} x_{l_2 m_2} + x_{l_1 m_1} y_{l_2 m_2}) C_{l_1 m_1 l_2 m_2}^{LM} \]

\[ \varphi_{Z_n}(t) = \varphi_{X_1}(a_1 t) \varphi_{X_2}(a_2 t) \ldots \varphi_{X_n}(a_n t) \]
Decomposition of CMB temperature fluctuations

\[ \Delta T(\hat{n}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{m=l} a_{lm} Y_{lm}(\hat{n}). \]

where \(a_{lm}\) are the spherical harmonic coefficients.

Gaussianity and reality of these fluctuations implies that real and imaginary part of spherical harmonic coefficients are mutually independent and both Gaussian.

\[ \begin{align*}
\langle x_{lm} \rangle &= \langle y_{lm} \rangle = 0, \\
\sigma^2(x_{lm}) &= \sigma^2(y_{lm}) = \frac{C_l}{2}. \quad \text{For } m \neq 0, \\
\langle x_{lm} \rangle &= 0, \\
\sigma^2(x_{lm}) &= C_l \quad \text{For } m = 0, \text{ imaginary part vanishes.}
\end{align*} \]
Combinations of random variables

\[ X_1 \sim N(0, \sigma_1^2) \]
\[ X_2 \sim N(0, \sigma_2^2) \]
\[ Z = X_1 X_2 \]
\[
 f_Z(z) = \frac{K_0 \left( \frac{|z|}{\sigma_1 \sigma_2} \right)}{\pi \sigma_1 \sigma_2}
\]

\[ (X_1, Y_1) \sim N(0, \sigma_1^2) \]
\[ (X_2, Y_2) \sim N(0, \sigma_2^2) \]
\[ Z = X_1 X_2 + Y_1 Y_2 \]
\[
 f_Z(z) \sim \text{Laplace}(0, 2/\sigma_1^2 \sigma_2^2)
\]

\[ X \sim N(0, \sigma^2) \]
\[ Y \sim N(0, \sigma^2) \]
\[ Z = X^2 - Y^2 \]
\[
 f_Z(z) \sim \frac{1}{2 \pi \sigma^2} K_0 \left( \frac{|z|}{2 \sigma^2} \right)
\]
Application of characteristic function approach

\[ l_1 = l_2, M = 0 \]

\[ A_{l_1 l_1}^{L0(R)} = \sum_{m_1(m_1>0)} (-1)^{m_1} 2(x_{l_1 m_1}^2 + y_{l_1 m_1}^2) C_{l_1 m_1 l_1}^{L0} + x_{l_1 0}^2 C_{l_1 0 l_1 0}^{L0} \]

\[ \chi^2 \text{ with } 2 \text{ d.o.f} \]

\[ \chi^2 \text{ with } 1 \text{ d.o.f} \]

type 1

type 2

\[ \varphi_{A_{l l}^{L0}}(t) = \prod \varphi_{\text{type } 1}(t) \varphi_{\text{type } 2}(t) \]
Recipe

- BipoSH coefficients are linear combination of some random variables.
  \[ A_{l_1l_2}^{JM} = \sum_{m_1m_2} a_{l_1m_1} a_{l_2m_2} C_{l_1m_1l_2m_2}^{JM} \]

- Find the characteristic function of each term present in linear sum.

- Assume NO non-linear correlation among terms, find characteristic function of these coefficients.
  \[ \varphi_{Z_n}(t) = \varphi_{X_1}(a_1t) \varphi_{X_2}(a_2t) \ldots \varphi_{X_n}(a_nt) \]

- Find Cumulant generating function from characteristic function.
  \[ g_Z(t) = \log[\varphi_Z(t)] \]

- Find Cumulants from cumulant generating function.
  \[ K_n = i^n g^n_Z(t) \bigg|_{t=0} \]

- Finally, find moments from Cumulants.
RESULTS!!

- Developed Faster code to calculate Bipolar coefficients (30x)
  - Can go up to high multipoles.

- Simulated Moments from 15000 Gaussian & isotropic realizations generated with best fit LCDM angular power spectrum.
BipoSH coefficients with $L = 0$ and $M = 0$

- Equivalent to CMB angular power spectrum.
- Well known result chi-square distribution.

$$A_{ll}^{00} = (-1)^l C_l$$

PDF, $l=6$  

PDF, $l=11$

Even multipoles – right skewed  

Odd multipoles – left skewed
BipoSH coefficients with $M = 0, L \neq 0$

- All terms in linear combination are independent of each other.
- Characteristic function for these coefficients is product of the characteristic function of each term in linear combination.
- Only BipoSH coefficients with Asymmetric Distribution!!

![Graphs showing Standard Deviation and Skewness](image-url)
These coefficients are always REAL.

Odd moments for these coefficients oscillate between positive and negative values for even and odd multipoles respectively.
BipoSH coefficients with $M \neq 0$

- Assumed independence among terms leads to mismatch between analytically derived moments and simulations.
- Terms in linear combination are linearly uncorrelated.
- Distribution is Symmetric, all odd moments vanishes.
- Account for non-linear correlations and simulation matches analytical moments.

\[ \bar{\mu}_n = \tilde{\mu}_n + \text{correction} \]
Conclusions

- Complete statistical information available for BipoSH coefficients with M=0.

- BipoSH coefficients with M=0 have Asymmetric distribution with even and odd multipoles being left and right skewed.

- Remaining coefficients have symmetric distribution.

- For coefficients with M not equal to zero, it turns out that terms in expansion are non-linearly correlated.

- To account these non-linear correlations, we supply a correction term to moments(up to kurtosis).

- These details need to be taken into account to have better assessment of any statistical isotropy violation detections in future data!!
THANK YOU