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Physical Significance of Planck Length

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The significance of Planck length in a quantum gravity model is investigated by concentrating on the conformal degree of freedom. It is shown that Planck length is a lower bound to physical proper length in any space-time. It is impossible to construct an apparatus which will measure length scales smaller than Planck length. These effects exist even in flat space-time because of vacuum fluctuations of gravity. It is shown that these fluctuations lead to a high energy cut-off in flat space field theories, thereby removing the divergence problem. The one-loop corrections to a self interacting scalar field is computed and shown to be finite.

I. INTRODUCTION AND SUMMARY

The three fundamental constants of physics G, $\hbar$, and c combine together to produce a natural length scale $(G\hbar/c^3)^{1/2}$. This scale $(\sim 10^{-33}\,\text{cm})$ is much farther removed from the scale of "elementary particles" $(\sim 10^{-13}\,\text{cm})$ than the particle physics scale from ordinary macroscopic physics scales $(\sim 10^{-2}\,\text{cm})$. It is well known that microscopic physics is fundamentally different from the deterministic laws of macroscopic physics. Pushing the analogy further many physicists have suggested that physics at Planck length may be fundamentally different from known microscopic particle physics. Such an interesting conjecture can be tested only when a theory for quantised gravity becomes available. On the other hand, any self-consistent model for quantum gravity must provide definitive answers regarding physics at Planck length.

The author has developed (along with J. V. Narlikar) an approach to quantum gravity which quantises the conformal factor of the metric, leaving the light cone structure as fixed (see Ref. [1, 2, 3]; for a detailed review, see Ref. [4]). This formalism is nonperturbative and produces quantum cosmological models which are free from singularities and horizon [5]. It is, therefore, of interest to see the results that emerge from this formalism as regards the physics at Planck length. We show in this paper that quantum gravity does change the fundamental concepts of physics rather drastically.

In Section II, we explore how conventional flat space-time physics is modified by quantum conformal fluctuations. By analysing an experiment to measure proper distances we show that quantum conformal fluctuations make length scales less

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than Planck length operationally ill-defined. It turns out that the fluctuations in the metric tensor make it impossible to define a unique proper length between any two events. For example, consider two events $(t, \mathbf{x})$ and $(t, \mathbf{y})$. We show that the probability for these two events to be seperated by a *proper* distance R is given by,

$$P(R) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left(-\frac{(R-R_0)^2}{2\sigma^2}\right), \tag{1}$$

where $R_0 = |\mathbf{x} - \mathbf{y}|$ and the width σ^2 is given by,

$$\sigma^2 = \frac{R_0^2}{4\pi^2}\left(\frac{L_p^2}{L^2}\right); \qquad L_p^2 = \frac{4\pi G\hbar}{c^3}. \tag{2}$$

Here L denotes the resolution limit of the experimental set up used to measure the proper length. In order to have a well-defined length $|\mathbf{x} - \mathbf{y}|$ between $(t, \mathbf{x})$ and $(t, \mathbf{y})$ we must have,

$$\sigma^2 \ll R_0^2 \Rightarrow L^2 \gg L_p^2. \tag{3}$$

Thus only when the length measurements are "coarse grained" over scales L much larger than L_p proper length has a clearcut meaning. Further, the same formalism can be used to show that the expectation value of the proper distance $l(x, y)$ between any two events x^i and y^i tends to a nonzero value as $x^i \to y^i$. That is,

$$\underset{x^i \to y^i}{\mathrm{Lt}}\ l^2(x, y) = L_p^2/4\pi^2. \tag{4}$$

Thus the proper length is bounded from below by a timelike interval $(L_p/2\pi)^2$.

In Section III, we apply these results to quantum theory of fields. We show that averaging over the small scale fluctuations of the metric removes the divergences from simple field theoretical models. The Green's functions of the free field theory remain finite at the coincidence limit. We calculate the oneloop contribution to the effective potential in a $\lambda\phi^4$ theory and show that it is finite. The self-energy contribution to the mass, etc., are explicitly calculable and will be finite. (For previous work along similar lines, see Refs. [11-13].)

In part IV, we speculate of the nature of geometry at Planck dimensions. It seems that our results indicate a stochastic spacetime with random fluctuations in the proper length.

II. Space-Time at Planck Length

2.1. *Flat Space as Gravitational Vacuum*

Classical general relativity identifies gravity with space-time curvature. In this picture, the proper (physical) interval between two events x^i and $x^i + dx^i$ is given by

$$ds^2 = g_{ik}(x)\, dx^i\, dx^k, \tag{5}$$

where $g_{ik}(x)$ are determined by Einstein's equations. In the absence of gravitational field, g_{ik} assumes the flat space-time values, $\eta_{ik} = \text{dia}(1, -1, -1, -1)$ and we get,

$$ds^2 = \eta_{ik}\, dx^i\, dx^k = dt^2 - dx^2 - dy^2 - dz^2. \tag{6}$$

It is also assumed, in a classical theory, that g_{ik} at any single event x^i can be measured with arbitrary accuracy. Thus the proper lengths in (5) and (6) can be measured (using suitable apparatus) as accurately as one wants. This immediately leads to the conclusion that,

$$\underset{x^i \to y^i}{\text{Lt}}\, ds^2 = 0, \tag{7}$$

where $y^i = (x^i + dx^i)$. This, rather trivial, result shows that proper interval between the events go to zero as the events approach each other.

Classical gravity, however, is only an approximation to quantum gravity. Thus flat space-time should be more properly considered to be the vacuum state of quantum gravity. The metric tensor g_{ik} becomes a quantum variable and is bedevilled by quantum fluctuations. It is well known that the fluctuations of quantum field will exist even in the vacuum state of the field. For example, one can compute the probability amplitude for measuring a magnetic field $\mathbf{B}(\mathbf{x}, t)$ in the vacuum state of quantum electrodynamics. The result is given by the well-known "ground state functional" [6, 7]:

$$\psi[\mathbf{B}(\mathbf{x}, t)] = N \exp\left(-\int \frac{\mathbf{B}(\mathbf{x}_1)\, \mathbf{B}(\mathbf{x}_2)\; d^3\mathbf{x}_1\, d^3\mathbf{x}_2}{16\hbar c\, |\mathbf{x}_1 - \mathbf{x}_2|^2} \right). \tag{8}$$

Just as electromagnetic field fluctuations exist in the electromagnetic vacuum, gravitational field fluctuations exist in the gravitational vacuum, viz. flat space-time. Since the gravitational field also describes the space-time geometry we must concede fluctuations of space-time geometry even in the flat space-time. Because of these fluctuations one can no longer talk about a unique value of the metric tensor at any event x^i. A more complicated, probabilistic description is required. In particular the fluctuations in g_{ik} will be divergent as the limit $x^i \to y^i$ is taken in (7), (5) and hence it is not clear, a priori, how result (7) would be modified. We shall now attempt to settle these questions in a simple model for quantum gravity.

2.2. *Quantum Conformal Fluctuations*

Considerable progress can be made in understanding the dynamics of the gravitational field if the attention is confined to the conformal degree of freedom of gravity (see references cited previously). Quantum gravity may be approached through the path integral ($\hbar = 1$),

$$K = \int \mathscr{D} g_{ik} \cdot \exp iS[g_{ik}], \tag{9}$$

where $(c=1)$,

$$S=\frac{1}{16\pi G}\int R\sqrt{-g}\,d^4x\equiv\frac{1}{12L_p^2}\int R\sqrt{-g}\,d^4x;\qquad L_p^2=\frac{4\pi G}{3}. \tag{10}$$

Most of the contributions to the path integral are expected to come from the classical solution, $g_{ik}=\bar{g}_{ik}$ (say). In considering the quantum *conformal* fluctuations, one evaluates the path integral in (9) over the class of metrics which are conformal to $\bar{g}_{ik}$. Such metrices may be represented in the form,

$$g_{ik}=(1+\phi(x))^2\,\bar{g}_{ik}, \tag{11}$$

where $\phi(x)$ is an arbitrary function of x^i. The zero value of ϕ corresponds to the classical solution $\bar{g}_{ik}$. Nonzero values of $\phi(x)$ give rise to nonclassical geometries. In terms of ϕ, the path integral becomes,

$$K=\int \mathscr{D}\phi\,\exp\left\{-\frac{i}{2L_p^2}\int\sqrt{-\bar{g}}\,d^4x\,[\phi^i\phi_i-\tfrac{1}{6}\bar{R}(1+\phi)^2]\right\}. \tag{12}$$

This can be evaluated in a closed form because of the quadratic nature of the path integral. Detailed discussion of this approach, the justification for concentrating on the conformal degree of freedom, etc., can be found in the references cited before, especially in [4], and will not be repeated here. In this paper, we shall take this formalism for granted and explore the consequences.

In particular, the above formalism can be used to answer the following question: What is the probability amplitude for the conformal fluctuations to have a given value $\phi(\mathbf{x})$ in the gravitational vacuum state (viz. flat space-time)? The answer is given by the vacuum functional for the conformal fluctuation:

$$\mathscr{P}[\phi(\mathbf{x}),t]=N\exp\left\{-\frac{1}{4\pi^2L_p^2}\int d^3\mathbf{x}\,d^3\mathbf{y}\,\frac{\nabla\phi(\mathbf{x})\cdot\nabla\phi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\right\}. \tag{13}$$

This expression gives the probability to observe a fluctuation $\phi(\mathbf{x})$ in the flat space-time. The time independence of reflects the fact that the ground state is a stationary state. The derivation of (13) proceeds exactly along the line of the derivation of (8); an interested reader will find the derivation in reference [5]. We shall now use this vacuum functional to study the quantum fluctuations of the metric around flat space.

2.3. *Quantum Fluctuations and Length Measurements*

If the space-time metric $\bar{g}_{ik}$ has a fixed value then the measurement of metric tensor will allow one to determine the proper distance between any two events. However, when the metric tensor is treated as a fluctuating quantum variable, it is necessary to measure the quantum field $\phi(x)$ before the geometrical features of the space-time can be determined.

It is well known that, in quantum field theory, the expectation value of objects like $\phi^2(x)$ are divergent. It is therefore necessary to approach the problem operationally. Let us consider a measurement of the field $\phi(\mathbf{x}, t)$ by an apparatus that has a spatial resolution limit of L (say). In other words, the apparatus does not distinguish $\mathbf{x}$ and $\mathbf{y}$ as different if $|\mathbf{x}-\mathbf{y}| < L$. (An ideal apparatus, of course, can be recovered in the limit of $L \to 0$). Thus the apparatus will actually measure the "smeared" (or "coarse grained") value of the field $\phi(\mathbf{x})$ averaged over a region $\sim L^3$. If we denote by $f(\mathbf{r})$ the sensitivity profile of the apparatus, then the "coarse grained" value is given by,

$$\phi_f(\mathbf{x}) \equiv \int \phi(\mathbf{x}+\mathbf{r}) f(\mathbf{r})\, d^3\mathbf{r}. \tag{14}$$

Once again, the ideal experiment has the sensitivity profile of a delta function $f(\mathbf{r}) = \delta(\mathbf{r})$, leading to $\phi_f(\mathbf{x}) = \phi(\mathbf{x})$. In general $f(\mathbf{r})$ will have a width of the order of L. That is $f(\mathbf{r})$ will be of the order of unity for $|\mathbf{r}| \ll L$ and will drop to zero rapidly for $|\mathbf{r}| \gtrsim L$. Thus (14) defines a "smeared" field $\phi_f(\mathbf{x})$ as an average over a region of size L^3.

We wish to emphasize that the need to consider smeared values of fields, as in (14), has *nothing* to do with gravity perse, and arises purely from the general formalism for quantum fields. (see, e.g., Ref. [8, p. 119]). Also the ideal experiment can always be recovered from this general case by taking $f(\mathbf{r}) = \delta(\mathbf{r})$, and thus nothing is lost in considering an arbitrary $f(\mathbf{r})$. In this discussion we have tacitly assumed that there exist measurements devices and procedures which will measure observables centred around some point $\mathbf{x}$. This assumption is always there in any discussion of physical observables; nevertheless, it *is* a non-trivial assumption. In fact even the coordinate label $\mathbf{x}$ implies that—classically, at least—some systems of rods and clocks have been set up in the space-time. Eventually one has to interpret the setting up of these devices itself as part of the measurement process. We shall ignore these additional complications in this paper and will assume that a classical background description exists, thereby providing an operational realisation of the coordinates like t, $\mathbf{x}$, etc.

The apparatus when used to measure ϕ_f will come up with (in general) different values in different trails. (This is to be expected because the vacuum state is *not* an eigenstate of field operators.) The quantity of interest is the probability *amplitude* for the measurement to give ϕ_f the value η (say). This can be easily seen to be given by,

$$\mathscr{A}\,[\phi_f = \eta] = \int \mathscr{D}\phi(\mathbf{x})\, \delta(\phi_f - \eta)\, \mathscr{P}[\phi(\mathbf{x})], \tag{15}$$

where $\mathscr{P}[\phi(\mathbf{x})]$ is the vacuum functional in (13). This expression can be evaluated in the fourier space in a straightforward manner. Writing,

$$\phi(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, q_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}} \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, (a_{\mathbf{k}} + i b_{\mathbf{k}})\, e^{i\mathbf{k}\cdot\mathbf{x}} \tag{16}$$

we can express (15) in terms of $q_{\mathbf{k}}$ by using,

$$\mathscr{P}[q_{\mathbf{k}}] = N \exp\left(-\frac{1}{2L_p^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} |\mathbf{k}| \, |q_{\mathbf{k}}|^2 \right) \tag{17}$$

and

$$\phi_f = \phi_f^* = \mathrm{Re} \int \frac{d^3\mathbf{k}}{(2\pi)^3} q_{\mathbf{k}} f_{-\mathbf{k}} \, e^{-i\mathbf{k}\cdot\mathbf{x}}$$

where we have defined,

$$f_{\mathbf{k}} = \int f(\mathbf{r}) \, e^{-i\mathbf{k}\cdot\mathbf{r}} \, d^3\mathbf{r}. \tag{18}$$

We get,

$$\mathscr{A} = \int \frac{d\lambda}{(2\pi)} e^{-i\lambda\eta} \prod_{\mathbf{k}} \int dq_{\mathbf{k}} \, N \exp \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left\{ -\frac{|\mathbf{k}|}{2L_p^2} |q_{\mathbf{k}}|^2 + i\lambda \ \mathrm{Re} \ (q_{\mathbf{k}} f_{-\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}) \right\}. \tag{19}$$

Treating the real and imaginary parts of $q_{\mathbf{k}}$ as independent, the Gaussian integrals in (19) for each $\mathbf{k}$ can be computed. This gives

$$\mathscr{A} = \int \frac{d\lambda}{2\pi} N' \exp\left[-i\lambda\eta - \frac{\lambda^2 L_p^2}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{|\mathbf{k}|} |f_{\mathbf{k}}|^2 \right]. \tag{20}$$

Doing the λ-integration and fixing the normalizing constant N' properly (by normalizing $\mathscr{A}$) we get the final result:

$$\mathscr{A}(\eta) = C \exp\left(-\frac{\eta^2}{4\Delta^2} \right) = \left(\frac{1}{2\pi\Delta^2} \right)^{1/4} \exp\left(-\frac{\eta^2}{4\Delta^2} \right), \tag{21}$$

where

$$\Delta^2 = L_p^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{|f(\mathbf{k})|^2}{2|\mathbf{k}|} \equiv \frac{L_p^2}{4\pi^2 L^2}. \tag{22}$$

We have defined in (22) the "resolution length" L for a distribution $f(\mathbf{r})$ by,

$$L^{-2} \equiv \frac{1}{4\pi} \int d^3\mathbf{k} \frac{|f(\mathbf{k})|^2}{|\mathbf{k}|}. \tag{22}$$

This definition is motivated by the fact that for a Gaussian $f(\mathbf{r})$ with a width σ,

$$f(\mathbf{r}) = \left(\frac{1}{2\pi\sigma^2} \right)^{3/2} \exp\left(-\frac{|\mathbf{r}|^2}{2\sigma^2} \right) \tag{23}$$

the resolution length L is equal to σ. Thus L essentially measures the width of $f(r)$ for any distribution. As long as one considers length measurements averaged over many Planck lengths (i.e., for $L \gg L_p$), Δ is almost zero and the probability in (20) is sharply peaked at $\eta = 0$. In this case, quantum fluctuations hardly affect the length measurements. However, as the resolution of the apparatus L goes to zero the fluctuations in η go on increasing. One can no longer talk about a unique proper distance between $(t, \mathbf{x})$ and $(t, \mathbf{y})$. When the conformal factor has a value η, the proper length—between $(t, \mathbf{x})$ and $(t, \mathbf{y})$—is given by

$$R^2 = (1+\eta)^2 R_0^2; \qquad R_0^2 = |\mathbf{x}-\mathbf{y}|^2. \tag{24}$$

Therefore the probability for the events $(t, \mathbf{x})$ and $(t, \mathbf{y})$ to be separated by a proper length R is given by,

$$P(R)dR = P(\eta(R))\frac{d\eta}{dR}\,dR. \tag{25}$$

Therefore,

$$P(R)\,dR = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left\{-\frac{(R-R_0)^2}{2\sigma^2}\right\} dR; \qquad R \geqslant 0 \tag{26}$$

with

$$\sigma^2 = \left(\frac{R_0^2}{L^2}\right)\left(\frac{L_p^2}{4\pi^2}\right); \qquad R_0 = |\mathbf{x}-\mathbf{y}|. \tag{27}$$

This expression (26) demonstrates the effect of quantum fluctuations in a neat manner. Because of the vacuum fluctuations of the metric, one cannot attribute a unique proper distance between two events $(t, \mathbf{x})$ and $(t, \mathbf{y})$. The *probability* that this proper distance has a value R (when the measurement is performed with a resolution L) is given by (26). The proper distance is peaked at the classical value R_0. To have any confidence in this R_0 we must have

$$\sigma^2 \ll R_0^2. \tag{28}$$

This, in turn, implies that

$$L^2 \gg L_p^2. \tag{29}$$

In other words, the concept of a unique distance between events ceases to have any meaning when $L \sim L_p$.

The previous analysis was confined to a space-like hypersurface of $t = \text{constant}$. We can trivially extend the analysis to four dimensions by taking a smeared value of $\phi(x)$ over a space-time region of "volume" L^4. Since we will consider the expectation values of proper distances in the next section, we will not bother to extend the above analysis now.

From the distribution (26) one gets the mean square value to be,

$$\langle R^2 \rangle = R_0^2 + \sigma^2 = R_0^2\left(1 + \frac{L_p^2}{4\pi^2 L^2}\right). \tag{31}$$

Clearly to measure a distance R_0 we need $L \leqslant R_0$. As the events approach one another $R_0 \to 0$, $L \to 0$ keeping $R_0 \geqslant L$. In this limit, we get, $\langle R^2 \rangle$ to be $\sim L_p^2$ if we assume $R_0 = L$ as R_0, $L \to 0$. In other words, we see that the fluctuations may lead to a lower bound to proper length. However, at this stage, L depends on the choice of $f(\mathbf{r})$ and the limiting value of (31) is not clear. We shall now consider this situation in more detail.

2.4. *Lower Bound to Proper Length*

Classically, there is nothing that prevents one from considering two events which are arbitrarily close. However, the analysis in the previous section clearly shows that such considerations may not have any physical relevance. In particular Eq. (20) can be written as an "uncertainty principle,"

$$\Delta\eta\, \Delta l \gtrsim L_p, \tag{32}$$

where Δl is the uncertainty in the measurement of proper distance and $\Delta\eta$ is the uncertainty in the conformal factor. [In our case, $\Delta l \sim L$ and $\Delta\eta$ is given by (23) to be (L_p/L)]. Therefore, the limit of zero proper separation ($\Delta l \to 0$) is operationally ill-defined.

Let us consider the expectation value of the line interval in the vacuum state:

$$\langle 0|ds^2|0\rangle = \langle 0|g_{ik}|0\rangle\, dx^i\, dx^k = (1 + \langle \phi^2(x)\rangle)\, \eta_{ik}\, dx^i\, dx^k. \tag{33}$$

However, it is well known that $\langle \phi^2 \rangle$ evaluated at a single event diverges. Also note that ds^2 involves for its definition two events x^i and $y^i \equiv x^i + dx^i$. Since we are interested in the limit $x^i \to y^i$, it is more proper to consider, (using the notation $l^2 = \eta_{ik}\, dx^i\, dx^k$) the limit,

$$\operatorname*{Lt}_{x\to y} \langle ds^2 \rangle \equiv \operatorname*{Lt}_{x\to y} \langle l^2(x, y)\rangle \equiv \operatorname*{Lt}_{x\to y} (1 + \langle \phi(x)\phi(y)\rangle)\, \eta_{ik}\, dx^i\, dx^k$$
$$\equiv \operatorname*{Lt}_{x\to y} (1 + \langle \phi(x)\phi(y)\rangle)\, l^2.$$

The "average" value (vacuum expectation value) is defined via the usual path integral formula,

$$\langle \phi(x)\phi(y)\rangle = \frac{\int \mathscr{D}\phi\, \phi(x)\phi(y) \exp i\mathscr{S}(\phi)}{\int \mathscr{D}\phi \exp i\mathscr{S}(\phi)}, \tag{35}$$

where the action $\mathscr{S}$ is given by (12) with $\bar{g}_{ik}=\eta_{ik}$; $\bar{R}=0$:

$$\mathscr{S}=-\frac{1}{2L_p^2}\int \phi^i\,\phi_i\,d^4x. \tag{36}$$

Defining,

$$Z[J]=\int \mathscr{D}\phi \exp\left[i\mathscr{S}+i\int J(x)\phi(x)\,dx\right] \tag{37}$$

we get

$$\langle\phi(x)\phi(y)\rangle=-\frac{1}{2}\frac{\delta^2 Z}{\delta J(x)\delta J(y)}\bigg|_{J=0}. \tag{38}$$

Straightforward calculation (noting the negative sign in (36)) gives,

$$\begin{aligned}\underset{x^i\to y^i}{\mathrm{Lt}}\langle l^2(x,y)\rangle &= \underset{i\to y^i}{\mathrm{Lt}}\{1+\langle\phi(x)\phi(y)\rangle\}\,l^2\\ &= \underset{x^i\to y^i}{\mathrm{Lt}}\langle\phi(x)\phi(y)\rangle\,l^2\\ &= \underset{x\to y}{\mathrm{Lt}}\frac{L_p^2}{4\pi^2}\cdot\frac{1}{(x-y)^2}\,l^2=\underset{x\to y}{\mathrm{Lt}}\frac{L_p^2}{4\pi^2}\frac{1}{l^2}\,l^2\\ &=\left(\frac{L_p^2}{4\pi^2}\right).\end{aligned} \tag{39}$$

In other words, the expectation value of the proper interval between any two events in the spacetime is bounded from below at $(L_p/2\pi)^2$. Quantum fluctuations produce a "residual length" just as zero point vibrations of an oscillator lead to a resiual ground state energy.

Before we proceed with the discussion of the important result (39) we would like to clarify a technical point. The fact that the conformal degree of freedom comes up with the wrong sign in kinetic energy term makes the *Euclidean* gravitational action unbounded from below. In the approach to quantum gravity described in Section 2.2., we do *not* consider the Euclidean section but work with the oscillating path integrals themselves. (We believe this approach is more physical because of the ambiguities involved in defining the Euclidean section for an arbitrary curved Riemannian spacetime.) In the Euclidean approach one first transforms from ϕ to another variable $\eta=i\phi$, obtaining,

$$\mathscr{S}[\eta]=\frac{1}{2L_p^2}\int \eta^k\,\eta_k\,d^4x. \tag{40}$$

Now changing to the imaginary time coordinate $\tau = it$ we get the Euclidean path integral,

$$K_E = \int \mathscr{D}\eta \exp\left(-\int \frac{d^4 x_E}{2L_p^2} \eta^k \eta_k\right). \tag{41}$$

This well-defined path integral will lead to the Euclidean Green's function and expectation value,

$$\langle \eta(x)\eta(x) \rangle_E = \frac{L_p^2}{4\pi^2} \frac{1}{(\tau_x - \tau_y)^2 + |\mathbf{x} - \mathbf{y}|^2} \equiv \frac{L_p^2}{4\pi^2} \frac{1}{(x-y)_E^2}. \tag{42}$$

Assuming (as is always done in Euclidean approach to quantum gravity) that $\langle \phi(x)\phi(y) \rangle_E$ can be obtained as analytic continuation from η, we get,

$$\begin{aligned} \langle \phi(x_E)\phi(y_E) \rangle &= -\langle \eta(x_E)\eta(y_E) \rangle \\ &= -\frac{L_p^2}{4\pi^2} \frac{1}{(x_E - y_E)^2}. \end{aligned} \tag{43}$$

Note that we are still working in the Euclidean spacetime $(\tau, \mathbf{r})$. The proper distance in terms of this Euclidean coordinates has the limiting value, (note that $dt^2 - d\mathbf{x}^2 = -(d\tau^2 + d\mathbf{x}^2)$)

$$\begin{aligned} \underset{x \to y}{\mathrm{Lt}}\ l^2(x, y) &= \underset{x_E \to y_E}{\mathrm{Lt}} \langle \phi(x_E)\phi(y_E) \rangle \{-(x_E - y_E)^2\} \\ &= \underset{x_E \to y_E}{\mathrm{Lt}} \left\{ -\frac{L_p^2}{4\pi^2} \frac{1}{(x_E - y_E)^2} \right\} \{-(x_E - y_E)^2\} \\ &= (L_p/2\pi)^2. \end{aligned} \tag{44}$$

Thus this procedure leads, in this particular case, to the same conclusion as obtained in (39) by straightforward means. We have gone through the arguments in the Euclidean space time so as to show that mathematical ambiguities, which are always present with oscillating path integral, do not introduce any spurious features into our discussion. Hereafter, we shall work with real or Euclidean spacetime depending on the convenience.

Coming back to our basic result (39) it is clear that the conformal degree of freedom plays a special role in quantum gravity. Mathematically speaking, two factors have gone in crucially into the result (39): (i) conformal factor multiplies the space-time line interval and (ii) conformal factor appears in the action with quadratic dependence. Physically, one may consider conformal factor as a "conjugate variable" to proper length. The vacuum fluctuations of this conformal degree of freedom produces a "zero-point distance" in a space-time.

Though we have been concentrating on the flat space-time the result (39) happens to be valid for an arbitrary space-time. In an arbitrary space-time, the Green's function that determine $\langle \phi(x)\phi(y) \rangle$ satisfy the equation,

$$(\Box + \tfrac{1}{6}R)G = 0. \tag{45}$$

However, the coincidence limit $(x \to y)$ of $G(x, y)$ again has the behaviour of s^{-2}, where s is the proper distance between x and y (see, e.g., Ref. [9]). This immediately leads to our result in (39).

Our conclusions in Section II can be summarized as follows: (i) Even in flat space-time calculations one must take into account the vacuum (conformal) fluctuations of gravity, (ii) the effect of conformal fluctuations is to introduce the lower bound, on the expectation value proper distance $\langle l^2(x, y) \rangle$:

$$\operatorname*{Lt}_{x \to y} \langle l^2(x, y) \rangle = (L_p/2\pi)^2. \tag{46}$$

We shall now consider the consequences of our result.

III. Consequences of the Finite Lower Bound

3.1. *Green's Functions in Field Theory*

The probability amplitude for a *non-relativistic* particle to propagate from $\mathbf{x}_1$ at t_1 to $\mathbf{x}_2$ at t_2 is given by the path integral,

$$K(\mathbf{x}_2 t_2; \mathbf{x}_1 t_1) = \int \mathscr{D}x(t) \exp i\mathscr{A}[x(t)], \tag{47}$$

where $\mathscr{A}$ is the nonrelativistic action

$$\mathscr{A} = \tfrac{1}{2}m \int \dot{\mathbf{x}}^2 \, dt. \tag{48}$$

In a similar vein, one may consider a relativistic propagator for a spinless particle to be a suitably defined path integral. Consider the amplitude, for a particle to propagate from x^i to y^i in an affine interval λ. This may be taken to be,

$$K(x, y; \lambda) = \int \mathscr{D}x(\lambda) \exp i\mathscr{A}[x], \tag{49}$$

where we take $\mathscr{A}$ to be,

$$\mathscr{A} = -\frac{m}{2}\int_0^\lambda ds \left[g_{ik} \frac{dx^i}{ds}\frac{dx^k}{ds} + 1 \right]. \tag{50}$$

(This, rather peculiar, way of writing $\mathscr{A}$ is completely equivalent to the more usual definition,

$$\mathscr{A} = -m \int ds \tag{51}$$

because of the identity $g_{ik}\,\dot{x}^i\dot{x}^k = 1$. However, this definition proves to be of convenience in what follows.) The probability amplitude for the particle to propagate from x to y, irrespective of the affine interval is given by,

$$\begin{aligned} G(x, y) &\equiv \int_0^\infty d\lambda K(x, y; \lambda) \\ &= \int_0^\infty d\lambda\, \mathscr{D}x \exp\left\{ -\frac{im}{2} \int_0^\lambda [g_{ik}\dot{x}^i\dot{x}^k + 1]\, ds \right\} \end{aligned} \tag{52}$$

In flat space ($g_{ik} = \eta_{ik}$) the path integral over the quadratic action can be performed giving, (assuming m to have a small negative imaginary part)

$$\bar{G}(x, y) \equiv \int_0^\infty d\lambda \left(\frac{m}{2\pi i\lambda}\right)^2 \exp -\frac{im}{2}\left\{\frac{(x-y)^2}{\lambda} + \lambda.\right\} \tag{53}$$

$$= \left(\frac{m}{2\pi i}\right)^2 \frac{1}{l} \int_0^\infty \frac{d\alpha}{\alpha^2} \exp -\frac{iml}{2}\left(\alpha + \frac{1}{\alpha}\right); \qquad (x-y)^2 = l^2$$

$$= -\frac{m^2}{2\pi^2}\frac{1}{l} K_1\,(iml); \qquad m = \operatorname*{Lt}_{\varepsilon \to 0}\,(m - i\varepsilon), \tag{54}$$

where we have used the integral representation of modified Bessel function,

$$K_\nu(x) = \int_0^\infty \frac{d\lambda}{2} \lambda^{\nu-1} \exp\left\{ -\frac{1}{2} x \left(\lambda + \frac{1}{\lambda}\right)\right\}. \tag{55}$$

It is easy to see that, the conventional scalar Green's function in field theory defined as,

$$G(x, y) \equiv -\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\varepsilon} \tag{56}$$

$$= -\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \int_0^\infty \frac{d\lambda}{i} \exp i\lambda(k^2 - m^2 + i\varepsilon)$$

$$= i\int_0^\infty d\lambda \exp -i\lambda(m^2 - i\varepsilon) \int \frac{d^4k}{(2\pi)^4} e^{i\lambda k^2 - ik\cdot(x-y)}$$

$$= \frac{1}{16\pi^2}\int_0 \frac{d\lambda}{\lambda^2} \exp -i\left[m^2\lambda + \frac{(x-y)^2}{4\lambda}\right]; \qquad m^2 = \operatorname*{Lt}_{\varepsilon\to 0}\,(m^2 - i\varepsilon)$$

$$= \frac{1}{4\pi^2}\frac{m}{l} K_1(iml) = -\frac{1}{2m}\bar{G}(x, y). \tag{57}$$

We thus see that the conventional field theory Green's function is same as that defined via the path integral in (52) except for the unimportant proportionality constant. Using (57) and (52) we write the field theory Green's function as,

$$G(x, y) = -\int_0^\infty \frac{d\lambda}{2m} \int \mathscr{D}x \left\{\exp -i\frac{m}{2}\int_0^\lambda [\dot{x}_i \dot{x}^i + 1]\, ds\right\}. \tag{58}$$

How does this Green's function change when we take the vacuum fluctuations into account? In particular, how does it behave as $x \to y$?

In the definitions given so far, it was assumed that the space-time remains flat and the proper distance between two events x, y remain at the classical value of $(x-y)^2$. In general, one should average over the vacuum (conformal) fluctuations. As we have seen previously this has the effect of replacing the classical value of the proper distance $(x-y)^2$ by,

$$\begin{aligned}\langle (x-y)^2 \rangle &= (1 + \langle \phi(x)\phi(y) \rangle)(x-y)^2 \\ &= (x-y)^2 + (L_p/2\pi)^2.\end{aligned} \tag{59}$$

Using (53) we get,

$$\langle G(x, y) \rangle = -\int_0^\infty \left(\frac{m}{2\pi i\lambda}\right)^2 \exp -\frac{im}{2}\left\{\frac{\langle (x-y)^2 \rangle}{\lambda} + \lambda\right\} \frac{d\lambda}{2m}. \tag{60}$$

In particular, the Green's function remains finite at the coincidence limit of $x \to y$

$$\begin{aligned}\underset{x \to y}{\text{Lt}} \langle G(x, y) \rangle &= \langle G(0) \rangle \\ &= -\int_0^\infty \frac{d\lambda}{2m}\left(\frac{m}{2\pi i\lambda}\right)^2 \exp -\frac{im}{2}\left\{\frac{(L_p/2\pi^2)}{\lambda} + \lambda\right\} \\ &= \frac{1}{2\pi}\frac{m}{L_p} K_2\left(i\frac{mL_p}{2\pi}\right).\end{aligned} \tag{61}$$

The modified Green's function in the Fourier space may be taken in the form, (using 60, 59),

$$\begin{aligned}\langle G(k) \rangle &\equiv \int d^4x\, e^{-ik\cdot x} \langle G(x) \rangle \\ &= -\int d^4x e^{-ikx} \int_0^\infty \frac{d\lambda}{2m}\left(\frac{m}{2\pi i\lambda}\right)^2 \exp -\frac{im}{2}\left[\frac{x^2 + (L_p/2\pi)^2}{\lambda} + \lambda\right] \\ &= -\int_0^\infty \frac{d\lambda}{2m}\left(\frac{m}{2\pi i\lambda}\right)^2 \exp -\frac{im}{2}\left(\frac{L_p^2}{4\pi^2} + \lambda\right)\left(\frac{2\pi\lambda}{im}\right)^2 \exp\left(-\frac{k^2\lambda}{2im}\right) \\ &= -\int_0^\infty \frac{d\lambda}{2m} \exp\left\{-\frac{imL_p^2}{8\pi^2\lambda} + \frac{i\lambda}{2m}(k^2 - m^2)\right\} \\ &= \frac{L_p}{2\pi i}\frac{1}{(k^2 - m^2 + i\varepsilon)^{1/2}} K_1\left(i\frac{L_p}{2\pi}(k^2 - m^2 + i\varepsilon)^{1/2}\right).\end{aligned} \tag{62}$$

The $(i\varepsilon)$ prescriptions are not necessary if we use the Euclidean four momentum k_{E} (so that $-k^2 = k_{\mathrm{E}}^2$). The Euclidean Green's function is,

$$\langle G(k_{\mathrm{E}})\rangle = \frac{L_p}{2\pi}\frac{1}{(k_{\mathrm{E}}^2+m^2)^{1/2}}\cdot K_1\left(\frac{L_p}{2\pi}(k_{\mathrm{E}}^2+m^2)^{1/2}\right). \tag{63}$$

The modefied Bessel function $K_1(z)$ has the asymptotic behaviours,

$$K_1(z) = \begin{cases} z^{-1}; & z \to 0 \\ \left(\dfrac{\pi}{2z}\right)^{1/2} e^{-z}; & z \to \infty \end{cases}$$

so that,

$$G(k_{\mathrm{E}}) = \begin{cases} (k_{\mathrm{E}}^2+m^2)^{-1} & (\text{as } L_p \to \infty) \\ \exp\left\{-\dfrac{L_p}{2\pi}(k_{\mathrm{E}}^2+m^2)^{1/2}\right\} & (\text{as } k_{\mathrm{E}} \to \infty). \end{cases} \tag{64}$$

Note that $\langle G(k_{\mathrm{E}})\rangle$ dies down exponentially for large k_{E} in contrast with the $G(k_{\mathrm{E}})$ which dies down only as k_{E}^{-2}. This feature is seen more transparently in the Green's function for massless field. Classically the Green's function is

$$D(x-y) = \frac{1}{4\pi^2 i}\frac{1}{(x-y)^2 + i\varepsilon}$$

which diverges as $x \to y$. The fluctuations replace this by

$$\begin{aligned}\langle D(x,y)\rangle &= \frac{1}{4\pi^2 i}\frac{1}{\langle (x-y)^2\rangle + i\varepsilon} \\ &= \frac{1}{4\pi^2 i}\frac{1}{(x-y)^2 + (L_p/2\pi)^2 + i\varepsilon}.\end{aligned}$$

As $(x \to y)$, we get the finite coincidence limit,

$$\langle D(0)\rangle = \frac{1}{iL_p^2}.$$

Mathematically the coincidence limit divergence of the Green's function originates from the high momentum behaviour of the integral in (56). Since the lower bound at Planck length acts as a high momentum cut-off, we get a finite value for the integrals. Physically, the propagator that describes the propagation from $(x \to y)$ does not recognize any path to have proper length below $(L_p^2/4\pi^2)$.

We wish to emphasize that we have *not* put a small distance cut-off *by hand.* This cut-off arises naturally because of the following reason: The Green's functions depend on the proper geodesic interval between the events. *However, the concept of a unique proper distance between the events does not exist when one recognizes the omnipresent vacuum fluctuations of gravity.* It is mandatory to average over the proper distances in various conformal geometries. We may indicate this process by the formula,

$$\langle G(x, y)\rangle \equiv \left\langle \frac{\int \mathscr{D}\psi\ \psi(x)\psi(y) \exp i\mathscr{A}[\psi]}{\int \mathscr{D}\psi(x) \exp i\mathscr{A}[\psi]} \right\rangle. \tag{65}$$

Here the path integral averaging produces the conventional Green's function that depends on the geodesic interval between x and y. The averaging $\langle\ \rangle$ averages over the proper distance and produces the final, physical, Green's function.

We conclude this section by commenting on a possible ambiguity in the above procedure as regards the averaging. So far we have obtained $\langle G(l)\rangle$ by taking $G(\langle l\rangle)$. One may argue that this prescription is ad hoc. If we average the Green's function over the geometries, or better still, average the final scattering cross section for a physical process over the geometries, the result may be different. Though such an ambiguity does exist, we defend our prescription on the following grounds: (i) proper distance is a clear cut physical entity which enters into the definition of various field theoretical objects. We feel that geometrical fluctuations should affect *only* the proper distance *directly.* The functions in field theory gets affected indirectly as in (63). (ii) We showed in Section II that vacuum gravitational fluctuations *do* affect the proper distance and leads to a nonzero lower bound. This alone is enough to produce a high energy cut-off. While the details of the process may depend on the averaging presciption, the essential result (finiteness at coincidence limit) does not. (iii) Later on we will speculate on a model for spacetime which is stochastic. The averaging procedure then arises in a natural fashion.

It is well known that divergences in the Green's function lead to divergences in the perturbation expansion in field theory. We shall now compute the one loop contribution to the effective potential in a scalar field theory and show that it is finite.

3.2. *Effective Potential for a Scalar Field*

Consider a quantum scalar field $\psi(x)$ described by an action,

$$\mathscr{A}_0 = \int dx\left(\tfrac{1}{2}\psi^i\psi_i - \tfrac{1}{2}m^2\psi^2 - V(\psi)\right) \equiv \mathscr{A}_0(\psi, V). \tag{66}$$

The quantum effective potential is defined via the relation, (see, e.g., Ref. [10])

$$\exp i\mathscr{A}_{\text{eff}}(\psi) = \frac{\int \mathscr{D}\eta \exp i\mathscr{A}_0(\psi+\eta, V)}{\int \mathscr{D}\eta \exp i\mathscr{A}_0(\psi+\eta, 0)}. \tag{67}$$

The path integrals can be evaluated in the one-loop approximation by retaining terms up to quadratic in η. This will give,

$$\exp i\mathscr{A}_{\text{eff}}(\psi) = \frac{\text{Det}^{-1/2}\,(\Box + m^2 + V''(\psi) - i\varepsilon)}{\text{Det}^{-1/2}\,(\Box + m^2 - i\varepsilon)} \tag{68}$$

leading to an effective potential [10, p. 207]

$$V_{\text{eff}} = \frac{1}{2}\int \frac{d^4k_{\text{E}}}{(2\pi)^4} \ln(1 + G(k_{\text{E}})\,V''), \tag{69}$$

where k_{E} is the Euclidean four momentum and $G(k_{\text{E}})$ is the Euclidean propagator,

$$G(k_{\text{E}}) = \frac{1}{k_{\text{E}}^2 + m^2} \tag{70}$$

Clearly the expression (69) is divergent at large k_{E}. Usually, one uses a cut off at large k_{E} and absorb the divergent parts into the parameters of $V(\psi)$.

The vacuum fluctuations of the conformal factor, as we have seen, replaces the free particle Green's function by $\langle G(k_{\text{E}})\rangle$, giving an effective potential,

$$\langle V_{\text{eff}}\rangle \equiv \frac{1}{2}\int \frac{d^4k_{\text{E}}}{(2\pi)^4} \ln(1 + \langle G(k_{\text{E}})\rangle V''). \tag{71}$$

We have found that,

$$\begin{aligned}\langle G(k_{\text{E}})\rangle &= +\frac{L_p}{2\pi}\frac{1}{(k_{\text{E}}^2+m^2)^{1/2}}\,K_1\left(\frac{L_p}{2\pi}(k_{\text{E}}^2+m^2)^{1/2}\right)\\ &\to +\frac{L_p}{2\pi}\left(\frac{\pi}{L_p}\right)^{1/2}\frac{1}{(k_{\text{E}}^2+m^2)^{3/4}}\\ &\quad\times\exp\left\{-\frac{L_p}{2\pi}(k_{\text{E}}^2+m^2)^{1/2}\right\}\qquad (k_{\text{E}}\to\infty).\end{aligned} \tag{72}$$

Clearly for large k_{E} the integrand in (71) dies down exponentially. This leads to a finite expression for $\langle V_{\text{eff}}\rangle$.

Because of the complicated form of $\langle G(k_{\text{E}})\rangle$ it is not possible to integrate (71) analytically. But it should be clear from (72) that replacing $G(k_{\text{E}})$ by $\langle G(k_{\text{E}})\rangle$ has the effect of a cut off at $\bar{k}_{\text{E}}(Lp_p/2) \cong 1$. Using this fact we can approximate (71) by,

$$V_{\text{eff}} \cong \frac{4\pi^4}{L_p^2}V'' + \frac{\pi^2}{2}(V'')^2\left[\ln\frac{V''L_p^2}{4\pi^2} - \frac{1}{2}\right] + O(mL_p) + \text{constant}. \tag{73}$$

For any realistic field theory (mL_p) will be an extremely small quantity and the approximation will be quite valid.

Once the basic feature presented in Section II is accepted these results need not come as any surprise. It is well known that a small distance cut-off will cure the divergence problem of field theory. We have only demonstrated this fact explicitly. Similar investigations in quantum electrodynamical processes are under way and will be published elsewhere.

IV: Some Futher Speculations

Since the idea of a lower bound on proper distance is of somewhat drastic nature, it is essential to examine the consequences of such an idea from various angles. Some possibilities are briefly discussed in this section.

In a space-time with the metric g_{ik}, the geodesic equation reads as,

$$\frac{d^2x^i}{ds^2} = -\Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds}. \tag{74}$$

We can show easily that, if $g_{ik} = \Omega^2 \eta_{ik}$, then,

$$\Gamma^i_{kl} = \frac{1}{\Omega}(\Omega_k \delta^i_l + \Omega_l \delta^i_k - \Omega^i \eta_{lk}). \tag{75}$$

Concentrating on the space components, we write,

$$\frac{dp^\mu}{ds} = -m\Gamma^\mu_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}. \tag{76}$$

If we make the further assumption that the fluctuations $\phi = \Omega - 1$ are small compared to one, then we can approximate eq. (76) to

$$\frac{d\mathbf{p}}{dt} = -mc^2\nabla\phi. \tag{77}$$

This equation lends itself to an interesting interpretation. In flat space a particle should feel no force; consequently the momentum will remain at a fixed value. However, the fluctuations of the conformal factor $\phi(\mathbf{x}, t)$ acts as a gravitational potential through (77). The right-hand side of (77) may be interpreted as a random, stochastic force acting on the "free" particle. Such a force makes the particle perform a random walk in the momentum space. After some time t, the mean value of momentum will remain zero; but the mean square value of the momentum increases in proportion to t. Let us make a rough estimate of this diffusion constant:

From our basic formula (13), we see that the mean square value of $\nabla\phi$ is given by,

$$\langle(\nabla\phi)^2\rangle \sim \frac{L_P^2}{l^4}, \tag{78}$$

where l is the characteristic scale over which measurements are made. (We expect $l \sim L$, resolution length, of Section I.) Thus the right hand side (77) may be approximated by a random force of magnitude

$$|\mathbf{f}| \sim mc^2 \cdot \frac{L_p}{l^2}. \tag{79}$$

Each "kick" due to this force produces a change in momentum of magnitude,

$$|\Delta p| \sim mc^2 \frac{L_p}{l^2} \cdot \Delta t \sim mc \left(\frac{L_p^2}{l^2}\right). \tag{80}$$

Standard analysis of random walk phenomenon will now give the probability $\mathscr{P}[\mathbf{p}, t]$ for the particle to have momentum $\mathbf{p}$ at time t is given by,

$$\mathscr{P}[\mathbf{p}, t] = \left(\frac{1}{2\pi D t}\right)^{1/2} \exp\left(-\frac{p^2}{2Dt}\right) \tag{81}$$

with the "diffusion coefficient,"

$$D = \frac{1}{2} (mc)^2 \left(\frac{L_p}{l}\right)^3 \left(\frac{c}{l}\right). \tag{82}$$

After time t, the mean square deviation in $\mathbf{p}$ is given by,

$$(\Delta \mathbf{p})^2 = Dt = \frac{1}{2} (mc)^2 \left(\frac{L_p}{l}\right) \left(\frac{ct}{l}\right). \tag{83}$$

Macroscopic measurements have $l \gg L_p$ and hence $\Delta \mathbf{p}$ will be far less than (mc). However, as $l \to L_p$ we note that the random fluctuations in $\Delta \mathbf{p}$ approaches the maximum value of mc. Thus the random fluctuations in the metric will prevent any possibility of measurements within $l \lesssim L_p$. The detailed implications of this idea are under investigation and will be published separately.

In the above discussion we have taken the "particles" to be the basic entities out of which spacetime is operationally constructed. One may take the alternative point of view and treat spacetime as fundamental. In that case, we should examine possible microstructures for spacetime that would allow the result,

$$\underset{dx^i \to 0}{\mathrm{Lt}}\ g_{ik}\, dx^i dx^k = \left(\frac{L_p}{2\pi}\right)^2. \tag{84}$$

One possibility is to model the spacetime with a "random lattice structure" in the following sense. Consider a (Euclidean) space-time in which the distance between any two events x^i and y^i can take all possible values r ($r > 0$) with a probability distribution,

$$p(r)dr = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left\{-\frac{(r - |\mathbf{x} - \mathbf{y}|)^2}{2\sigma^2}\right\} dr. \tag{85}$$

In such a space-time,

$$\langle r^2 \rangle = |\mathbf{x} - \mathbf{y}|^2 + \sigma^2 \tag{86}$$

and by taking $\sigma = (L_p/2\pi)$ we can recover the previous results. In such a formalism one has to carefully average over all possible distances r as well.

We feel that the results of this paper establish a prima facie case for the role of Planck length as the ultimate lower bound in the length scales. Only further investigation can show whether the result is of fundamental significance or yet another red herring.

Appendix

The major results of Section III arise form replacing the average value of the Green's function $\langle G(l) \rangle$ by $G(\langle l \rangle)$. The results can be placed on a firmer footing if one can show that functional averaging over conformal degree of freedom replaces $G(l)$ by $G(\langle l \rangle)$, as $l \to 0$.

For an arbitrary system of fields one cannot evaluate the functional integral over conformal degrees of freedom exactly. As can be seen from the expansion,

$$\frac{1}{l^2 + L_p^2} = \frac{1}{l^2} - \frac{L_p^2}{l^4} + \cdots \tag{A.1}$$

perturbation to finite order in L_p^2 *will not* give finite value for $G(l)$ as $l \to 0$. Thus one requires a nonperturbative approximation of the functional average over conformal factor.

There is one simple case in which we can evaluate this integral exactly. Consider a scalar field $B(x)$ described by the conformally invariant action,

$$\mathscr{A} = \frac{1}{2}\int \left(B^i B_i - \frac{1}{6} R B^2\right) \sqrt{-g}\, d^4x. \tag{A.2}$$

Let $G(x, y)$ be the greens function for $B(x)$, i.e.,

$$G(x, y) = \int \mathscr{D}B\, B(x) B(y) \exp i\mathscr{A}[B]. \tag{A.3}$$

It is well known that under the conformal transformations $g_{ik} \to \Omega^2(x) g_{ik}$, G transforms as,

$$G \to G(x, y; \Omega) \equiv \Omega^{-1}(x)\, \Omega^{-1}(y)\, G(x, y). \tag{A.4}$$

Therefore, in any conformally flat space-time (with $g_{ik}=\Omega^2(x)\eta_{ik}\equiv(1+\phi(x))^2\eta_{ik}$) the Green's function G can be written as,

$$G(x,y;\Omega)=\Omega^{-1}(x)\,\Omega^{-1}(y)\,G_{\text{flat}}$$
$$=\frac{1}{[1+\phi(x)]}\frac{1}{[1+\phi(y)]}\frac{1}{4\pi^2 i}\frac{1}{(x-y)^2}. \qquad \text{(A.5)}$$

To consider the effect of quantum conformal fluctuations we have to 'average' (A.5) over all $\phi(x)$. That is, we have to compute

$$\langle G(x,y)\rangle\equiv\int\mathscr{D}\phi G(x,y;\Omega)\exp i\mathscr{A}[\phi]$$
$$=\int\mathscr{D}\phi G(x,y;1+\phi)\exp\left\{-\frac{i}{2L_p^2}\int\phi_i\phi^i\alpha^4x\right\}. \qquad \text{(A.6)}$$

To perform these functional integrals we shall use the parametrization

$$\frac{1}{\Omega(x)\Omega(y)}=i\int_0^\infty d\lambda\exp i\lambda\Omega(x)\Omega(y). \qquad \text{(A.7)}$$

Then we can write

$$\langle G(x,y)\rangle=i\int_0^\infty d\lambda\int\mathscr{D}\phi\frac{\exp\left\{i\lambda(1+\phi(x))(1+\phi(y))-\frac{i}{2L_p^2}\int\phi_i\phi^i d^4x\right\}}{4\pi^2 i(x-y)^2}$$
$$=\frac{i}{4\pi^2}\frac{1}{i(x-y)^2}\int_0^\infty d\lambda\langle\exp i\lambda(1+\phi(x))(1+\phi(y))\rangle$$
$$=\frac{1}{4\pi^2 i}\frac{1}{(x-y)^2}\frac{1}{[1+\mathscr{G}(x,y)]}, \qquad \text{(A.8)}$$

where,

$$\mathscr{G}(x,y)\equiv\langle\phi(x)\phi(y)\rangle=\int\mathscr{D}\phi\,\phi(x)\phi(y)\exp-\frac{i}{2L_p^2}\int\phi^i\phi_i d^4x$$
$$=\frac{L_p^2}{4\pi^2}\frac{1}{(x-y)^2}. \qquad \text{(A.9)}$$

We thereby obtain

$$\langle G(x,y)\rangle=\frac{1}{4\pi^2 i}\frac{1}{(x-y)^2+(L_p/2\pi)^2} \qquad \text{(A.10)}$$

proving $\langle G(l)\rangle=G(\langle l\rangle)$.

The "proof" above crucially depends on the parametrization used in (A.7) and the admissibility of interchange of various integrations. As it stands (A.7) is ill-defined; however, one can always construct more elaborate regularization procedures which will ultimately ensure the equality:

$$\left\langle \frac{1}{[1+\phi(x)][1+\phi(y)]} \right\rangle = \frac{1}{1+\mathscr{G}(x,y)}. \tag{A.11}$$

These ambiguities, added to the fact that the scalar field B has a special coupling $(1/6RB^2)$, makes the status of derivation dubious.

We feel that the result $\langle G(l)\rangle = G(\langle l\rangle)$ is much more fundamental and general than as indicated by the derivation. It is most likely to be related to some stochastic property of space-time itself.

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